On the category of metric compact Hausdorff spaces

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A category

- Separated metric compact Hausdorff spaces (compact Hausdorff space + metric).

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- Nachbin spaces (compact Hausdorff space + order).

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Metric is similar to Order (quantale-enrichment).

(Quantales are special symmetric strict monoidal categories.)

Historical Background

Compact Hausdorff spaces = Eilenberg-Moore algebras for the ultrafilter monad on **Set**.

CompHaus is a variety of infinitary algebras; it has an algebraic flavor. For example, it is Barr-exact.

CompHaus has also a topological flavor.

Usually, the opposite of a category with a topological flavor has an algebraic flavor:

- While **Top** is not regular, $\mathbf{Top}^{\mathrm{op}}$ is regular (in fact, a quasivariety of infinitary algebras [Barr, Pedicchio, 1995]).
- The opposite of the category of Stone spaces is a variety of algebras (Boolean algebras) [Stone, 1936]:

  $$\hom_{\mathbf{Stone}}(-, \{0,1\}) \colon \mathbf{Stone}^{\mathrm{op}} \to \mathbf{Set}$$

  is monadic.
  (We recall that every monadic functor to **Set** is representable, by the free algebra on one generator.)

Theorem ([Duskin, 1969], all details in [Barr, Wells, 1985])

The functor

$$\hom_{\mathbf{CompHaus}}(-, [0,1]) \colon \mathbf{CompHaus}^{\mathrm{op}} \to \mathbf{Set}$$

is monadic.

$\mathbf{CompHaus}^{\mathrm{op}}$ is a variety of infinitary algebras: it has an algebraic flavor.

CompHaus is complete, cocomplete, and Barr-coexact, $[0,1]$ is regular injective and a regular cogenerator ($=$ Urysohn's lemma).

Topology + *order*

Stone duality (between Stone spaces and Boolean algebras) has an important generalization to ordered-topological spaces: Priestley duality.

Priestley space := Stone space + compatible partial order.

Priestley duality [Priestley, 1970]: Priestley spaces are dual to bounded distributive lattices (which form a variety).

$$\mathrm{hom}_{\mathbf{Priestley}}(-, \{0,1\}) \colon \mathbf{Priestley}^{\mathrm{op}} \to \mathbf{Set}$$

is monadic.

Compact Hausdorff	Stone
$\mathbf{CompHaus}^{\mathrm{op}} \xrightarrow{\mathrm{hom}(-,[0,1])} \mathbf{Set}$	$\mathbf{Stone}^{\mathrm{op}} \xrightarrow{\mathrm{hom}(-,\{0,1\})} \mathbf{Set}$
Compact Hausdorff + order	Stone + order
?	$\mathbf{Priestley}^{\mathrm{op}} \xrightarrow{\mathrm{hom}(-,\{0,1\})} \mathbf{Set}$

Question [Hofmann, Neves, Nora, 2018]: is there an analogue with "*compact Hausdorff* spaces + order" instead of "*Stone* spaces + order"?

Nachbin space (a.k.a. compact ordered space) [Nachbin, 1948]: compact Hausdorff space + compatible partial order.

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Theorem (A., 2019)

$$\mathrm{hom}_{\mathbf{Nachbin}}(-,[0,1])\colon \mathbf{Nachbin}^{\mathrm{op}} \to \mathbf{Set}$$

is monadic.

(See [A., Reggio, 2019] for a nicer proof.)

It is also a way to collect categorical properties of **Nachbin**:

1. **Nachbin** is complete and cocomplete [Tholen, 2009].
2. $[0, 1]$ is a regular injective regular cogenerator [Nachbin, 1960].
3. **Nachbin** is Barr-coexact [A., Reggio, 2020]. (Coregularity and something more already in [Hofmann, Neves, Nora, 2018]).

Separated metric compact Hausdorff spaces

From Lawvere, we know that order is similar to metric.

Is there an analogue in the metric setting?

Before recalling separated metric compact Hausdorff spaces, let us see some drawbacks of the category of classical compact metric spaces and non-expansive maps:

- **not cocomplete**.
  Remedy: allow distance ∞.
- **not complete**.
  Remedy: topology **compatible** with the metric, rather than **induced** by it.

Definition

A *metric* on a set X is a map $d\colon X \times X \to [0, \infty]$ satisfying:

- (reflexivity) $d(x, x) = 0$;
- (triangle inequality) $d(x, z) \leq d(x, y) + d(y, z)$.

A metric is *separated* if $d(x, y) = 0 = d(y, x)$ implies $x = y$.

All results are true also when restricted to the symmetric case $(d(x, y) = d(y, x).)$

If d is only allowed to take values 0 and ∞,

- metric = preorder (where $d(x, y) = 0$ means $x \leq y$),
- separated metric = partial order.

Definition ([Hofmann, Reis, 2018])

(Separated) metric compact Hausdorff space $:=$ compact Hausdorff space X equipped with a lower semicontinuous (separated) metric $X \times X \to [0, \infty]$.

Lower semicontinuous:

$$d(x_0, y_0) \leq \liminf_{\substack{x \to x_0 \\ y \to y_0}} d(x, y).$$

I.e.: small topological perturbations may yield great increments in distances, but not great decrements.

Equivalently, continuous wrt the topology generated by the sets $(a, \infty]$.

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Example: any compact metric space (in the classical sense).

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Example: $[0, \infty]$, with $d(a, b) = \begin{cases} b - a & \text{if } a < b; \\ 0 & \text{otherwise.} \end{cases}$

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Example: For any set X, $[0, 1]^X$ with the product metric (= sup metric) and the product topology.

Metric compact Hausdorff spaces are the algebras for the "metric ultrafilter" monad on the category of metric spaces and nonexpansive maps (See [Hofmann, Reis, 2018], building on [Tholen, 2009]).

$\mathbf{MetCH}_{\mathrm{sep}}$:= category of separated metric compact Hausdorff spaces and nonexpansive continuous maps.

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Question

Is

$$\hom_{\mathbf{MetCH}_{\mathrm{sep}}}(-, [0, \infty]) \colon \mathbf{MetCH}_{\mathrm{sep}}^{\mathrm{op}} \to \mathbf{Set}$$

monadic?

I.e.:

1. Is $\mathbf{MetCH}_{\mathrm{sep}}$ complete and cocomplete?

2. Is $\mathbf{MetCH}_{\mathrm{sep}}$ Barr-coexact?

3. Is $[0, \infty]$ a regular injective regular cogenerator of $\mathbf{MetCH}_{\mathrm{sep}}$?

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   ? (*Future work.*)

Theorem ([A., Hofmann, 2025])

1. **MetCH**$_{\mathrm{sep}}$ *has (epi, regular mono) factorization:*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, dashed, two heads, "\text{surjection}"'] & & Y \\ & f[X] \arrow[ur, dashed, hook, "\text{embedding}"'] & \end{tikzcd}$$

equip $f[X]$ with the metric and topology induced by Y

2. *epis = surjective morphisms;*
3. *regular monos = strong monos = embeddings.*

Theorem ([A., Hofmann, 2025])

$\mathbf{MetCH}_{\mathrm{sep}}$ *is <u>co</u>regular.*

$$
\begin{tikzcd}
K \arrow[r, hook] \arrow[d, "f"'] & X \arrow[d, dashed] \\
K' \arrow[r, hook, dashed] \arrow[ur, phantom, "\ulcorner", very near end] & X'
\end{tikzcd}
$$

Given an embedding $K \hookrightarrow X$ and a morphism $f\colon K \to K'$, inside X we can replace K by a copy of K' (and appropriate adjustments outside of K' induced by f).

Theorem ([A., Hofmann, 2025])

$\mathbf{MetCH}_{\mathrm{sep}}$ *is Barr-coexact.*

For a separated metric compact Hausdorff space X, a closed subset $K \subseteq X$ induces a quotient of $X + X$ by gluing the two copies of K.

Barr-coexactness: every surjective morphism $X + X \twoheadrightarrow Z$ satisfying coreflexivity, cosymmetry, cotransitivity (first-order conditions) arises in this way.

To sum up

$\mathbf{MetCH}_{\mathrm{sep}}$:= category of separated metric compact Hausdorff spaces and continuous non-expansive maps. [Hofmann, Reis, 2019]

Is

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1. Is $\mathbf{MetCH}_{\mathrm{sep}}$ complete and cocomplete? ✓ [Tholen, 2009]
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Thank you.